Reconstruction of Compressively Sensed images using Regularized Sparse Dictionary Learning and Adaptive Spectral Filtering

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***Abstract*— Sparse representation using over-complete dictionaries have shown to produce good quality results in various image processing tasks. Dictionary learning algorithms have made it possible to engineer data-adaptive dictionaries that have promising applications in image compression and image enhancement. The most common sparse dictionary learning algorithms use the techniques of matching pursuit and K-SVD iteratively for sparse coding and dictionary learning respectively. While this technique produces good results, it requires a large number of iterations to converge to an optimal solution. In this article, we use a closed-form stabilized convex optimization technique for both sparse coding and dictionary learning. The approach results in providing the best possible dictionary and the sparsest representation resulting in minimum reconstruction error. We have used the proposed algorithm for compressed sensing of satellite images. Once the image is reconstructed from the compressively sensed samples, we use adaptive spatial and frequency domain filtering techniques to move towards exact image recovery. It is seen from the results that the proposed algorithm provides much better reconstruction results than conventional sparse dictionary techniques for a fixed number of iterations. Depending upon the number of details present in the image, the proposed algorithm is seen to reach the optimal solution with a significantly lower number of iterations. Consequently, high PSNR and low MSE is obtained using the proposed algorithm for our compressive sensing framework.**

Keywords*— *sparse representation; dictionary learning; compressed sensing; Tikhonov Regularization; Ridge Regression, Convex Optimization.

I. Introduction

The classic Nyquist-Whittaker-Shannon sampling theorem dictates us to sample at least twice as many samples per unit time of a signal as the maximum frequency content of the signal to reconstruct it efficiently from its sampled version. However, if we exploit the structural properties of the signal, such as sparsity, then we can get away with fewer samples to achieve complete reconstruction. This is the basis of Compressed Sensing. The tools of Compressed Sensing are of use if the signal possesses some structure as mentioned above. If the signal original doesn't have such a structure, say sparsity, then we can induce it by transforming the signal onto some appropriate bases, such as the Wavelet bases. Some bases such as the Wavelet bases are empirically known to yield sparse representations of the signals. Thus, the estimation problem (often a linear inverse problem) can then be solved for finding the coefficients of the base's transformation than the elements of the signal. Rather than using such empirical dictionaries, we can go a step further to learn the bases dictionary from the signal itself, such that, on this basis, the signal has the sparsest representation. The technique of Sparse Dictionary Learning, helps us achieve this. Thus, while the established dictionaries provide universal bases to represent data, the resulting sparse representations may have lesser sparsity than those obtained using data-adaptive dictionaries. In either case, over-complete dictionaries need to be used so that data may be efficiently represented sparsely.

Conventional Sparse Dictionary learning techniques employ the Matching Pursuit algorithm for Sparse Coding and K-SVD algorithm for data-adaptive Dictionary Learning [2][3][8]. The technique involves iterative execution of Sparse Coding and Dictionary Update to obtain the optimal solution in terms of sparsity and reconstruction error. The number of iterations can be controlled explicitly or by an error metric such as MSE. Several different techniques have been employed to implement sparse dictionary learning systems [5] [6]. The sparse dictionary learning technique has also been applied to compressed sensing over the past decade [4] [5] [6] [7]. Some literature on Compressed Sensing introduces the idea of compressive sensing as a multiplication of a fat matrix with the signal to be sensed. Thus, an undetermined system is considered, to which the signal is fed as an input and a lower-dimensional signal is produced as an output that is processed in the pipeline. In this case, however, a sensing system characterized by the sensing matrix has to have the complete incidence of the signal for compressive sensing. We go a step further and consider a sparse sensing matrix that is rank deficient. Thus, we get away with sensing only some parts of the signal incident on the sensing system. This intuitively refers to involving a dual compressive stage in the compressed sensing pipeline.

In this article, we use a more optimized technique for Sparse Dictionary Learning built on the Convex Optimization framework with Tikhonov Regularization and complement it with an efficient image recovery scheme to reconstruct compressively sensed satellite images without assuming any structure in the sensed images.

II. Methodology

In the proposed framework, we begin the Sparse Dictionary Learning process by randomly initializing a dictionary. We use the random initialization approach to reduce the time consumption of the initialization process as opposed to that resulting from the use of other initialization techniques like clustering. As mentioned earlier, the Sparse Dictionary Learning process can be divided into 2 stages which are executed iteratively, viz. Sparse Coding & Dictionary Update.

The algorithm uses an l2-regularized convex optimization technique for both sparse coding and dictionary update. Since this involves solving a linear inverse problem for sometimes rank deficient underdetermined system and overdetermined systems respectively, we use Tikhonov Regularization to stabilize the optimization to noise. A closed-form solution of Tikhonov Regularization, also known as Ridge Regression is used to find the optimal solution to our hypothesis. The technique is a logical extension to the conventional methods used for Sparse Dictionary Learning. The basic idea of sparse coding is based on identifying atoms and calculating the respective weights such that their linear sum is closely similar to the original data point. We approach this as a Stabilized Least Squares (Convex) optimization problem trying to reduce the mean square error of the original and the reconstructed signal. Thus, using Normal Equation we find the optimal solution for sparse representation which produces the least reconstruction error.

Conventional techniques such as K-SVD aim to incorporate an error vector as a representative vector for the data so that reconstruction error is minimized. We look at this from the MSE optimization point of view. The following sections discuss the involved ideas in more detail.

A. *Sparse Coding*

Sparse Coding of data is done using a dictionary. The dictionary may either be an established basis like DCT or may be learned from the data. Conventional techniques use techniques like Projection Pursuit, Basis Pursuit, and Matching Pursuit to achieve sparse coding. In our algorithm, we solve a convex optimization problem for sparse coding.

For every data point, we begin by finding the most appropriate weight for an atom in the dictionary. For this, we assume that the given data points can be effectively represented using only one dictionary atom. We then find the best weights for a given signal for all the corresponding atoms of the dictionary using Tikhonov Regularization. Our sparse coding system uses an adaptive sparsity scheme which allows the system to set up to L (user-defined) non-zero values in the data point's sparse representation. Using the method mentioned above, after we calculate the best 1-sparse representation of the signal with the given dictionary, we fix this atom and go on to find the next best atom in the dictionary. This is again found by choosing the weights which produce the least reconstruction error with the original signal. The weights are calculated using Ridge Regression as mentioned above. Note that, when a new atom is being chosen to be included in the signal's sparse representation, the weights associated with the previously chosen best atoms are changed so as to incorporate the contribution of the new atom. Also, note that we move from higher sparsity to lower sparsity up to L-sparse representation only if the reconstruction error reduces by lowering the sparsity.

B. *Dictionary Update*

The dictionary is updated after every sparse coding stage. As with other dictionary update techniques like K-SVD, the atoms in the dictionary are updated one-by-one using the data points which use those atoms in their sparse representation.

For every atom in the dictionary, we first find the signals which use that atom in their sparse representation. At this stage, we have the original signal and their sparse representations using that atom. With the available weights in the sparse representation, we again solve a convex optimization problem to find the best atom which will reduce the MSE of the dataset using that atom, using the Ridge Regression technique. This is then repeated for all the atoms in the dictionary. This completes the dictionary update stage which is then followed by the next cycle of sparse coding.

C. *Compressed Sensing*

Assume that the signal to be sensed compressively is $x \subset R^{N \times M}$. In the compressive image sensing context established above, where each column of x is an unrolled block of the image. Also, suppose that there exists a sparse representation of the signal as –

$$x = D\alpha \tag{1}$$

where $D \subset R^{N \times L}$ is an appropriate dictionary (over-complete) and $\alpha \subset R^{L \times M}$ is the corresponding appropriate sparse representation, one for each image block. Note again that both D and α are unknown here at this stage. If we multiply the above system with a deterministic fat matrix Q (called the sensing matrix), such that $Q \subset R^{K \times N}$ where $K << N$, then we have –

$$x_{cs} = Qx = QD\alpha \tag{2}$$

such that $x_{cs} \subset R^{K \times M}$ thus having a dimension less than x. Note here that the sensing matrix is a part of the compressed sensing system. We thus sense x_{cs} instead of x, and thus (compressively) sense less. Now if we apply our Sparse Dictionary Learning technique, introduced above, to represent x_{cs} we have –

$$x_{cs} = D_{cs}\alpha_{cs} \tag{3}$$

where D_{cs} and α_{cs} have appropriate dimensions. Note here again that we have D_{cs} and α_{cs} in hand, as opposed to D and α previously. If we impose that $\alpha = \alpha_{cs}$ indeed, then we have $D_{cs} = QD$ so that $D = Q^{\dagger}D_{cs}$ where † denotes the Moone-Penrose Inverse. Thus, we recover x from x_{cs} as –

$$x_{rec} = Q^{\dagger}D_{cs}\alpha_{cs} = D\alpha \tag{4}$$

This is our proposed Compressed Sensing Framework using Convex Sparse Dictionary Learning.

D. Image Recovery and Enhancement

As mentioned above, the sparse dictionary learning algorithm used above iteratively solving for the dictionary and the sparse coding iteratively, thus performing least squares estimation for overdetermined and underdetermined systems respectively, due to the dimensionalities of the system structures (matrices) involved in the calculation.

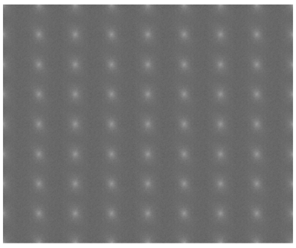


Fig. 1. Spectrum of the Compressively Sensed Image

Since the systems are not often full rank, they introduce null space error which propagates through the iterations. Thus, when the recovered signal x_{rec} is assembled into an image by performing the reverse of the unrolling operation, we observe periodic noise in the image, as seen from the center-shifted frequency spectrum (Fig. 1) of the recovered image. The desired noise-free signal viz. x lies at the center of the spectrum. Thus, we create an elliptical filter mask (Fig. 2) for such an image automatically by processing the spectrum of the recovered image by using the noise peaks. The desired image is then recovered through an Inverse Fourier Transform after above Spectral Filtering followed by an appropriate Gamma Transformation.

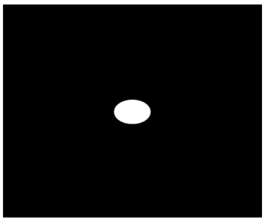


Fig. 2. The Adaptive Spectral Mask

III. Convex Optimization By Tikhonov Regularization

Ridge Regression [10] is used to solve an optimization problem when the cost function is convex such as the Mean Squared Error (MSE), Cross-Entropy, etc. for perturbation induced linear inverse problems with rank deficient systems. Since the cost function (MSE here) is parabolic, it is convex and has only one optimum. The Tikhonov Regularization is essentially the closed-form solution of the MSE cost function optimum.

$$w_{tikhonov} = (X^T X + \lambda I)^{-1} X^T y \qquad (5)$$

The (5) obtained is known as the Normal Equation and hence represents the optimal parameter solution for a convex optimization problem. In our algorithm, during sparse coding, we repeatedly calculate the best w for given X and y and then choose the best atom and its respective w for sparse representation. Similarly, we use the dictionary update stage where the atom is considered to be the model parameter and the sparse coded weights represent X with the original signal being y. The use of Normal Equation thus helps us obtain the best possible solution in terms of MSE and sparsity at every cycle of sparse representation and dictionary update.

The Tikhonov Regularization effectively truncates the part of the SVD which is weighted by small singular values of the system making its output stabilized in the presence of noise. This is highly essential due to channel noise during sensing.

IV. Results

To study the performance characteristics of the proposed algorithm and to compare it with other conventional techniques, we consider a dataset with 100 points each represented by 10 features, consistent throughout the study. Tables I and II show the results of our proposed algorithm using a complete dictionary for lambda equal to zero and compare it with other conventional techniques. It is seen that our algorithm provides the best results in terms of PSNR and MSE. Table II provides the results of the same test setup after 50 iterations. It is still observed that our proposed algorithm provides the best results. One can also observe that in a single iteration of our algorithm, we reach near the optimal solution unlike other techniques.

It is a well-known empirical fact that using over-complete dictionary results in better reconstruction results. However, when the algorithm was run for an over-complete dictionary, a very small difference in performance in terms of PSNR and MSE was observed. Thus, depending on whether space or time is a concern, one may choose to set the dictionary size appropriately. If storage space is to be minimized, our algorithm has to be run for a larger number of iterations using a complete dictionary. If the minimization of run-time is of primacy, then an over-complete dictionary may be used which would require much fewer iterations to produce satisfactory results.

It is observed that for our algorithm that as the sparsity reduces the PSNR improves. It may be reasoned based on the way dictionary update and sparse coding is done using our algorithm. In the first cycle, using a particular dictionary, we find the quasi-optimal weights per atom in the sparse coding stage resulting in a 1-sparse representation of the original signal. Since, the weights are already optimal for the given iteration, when a new atom is chosen to approximate the original signal considering the previous atom to be fixed, the approximation error increases. Thus, the sparse coding technique using our algorithm is found to normally converge on the 1-sparse representation of the original signal. In the following stage, we find the quasi-optimal atoms in complement with the quasi-optimal weights which results in an almost optimized solution in the first cycle of the algorithm itself. We now apply our algorithm for compressed sensing of a satellite image (Fig. 3(a)) of dimension 2800 x 3400. The image is divided into non-overlapping equal size blocks of size 8x8 with a sensing matrix having 4 rows. A dictionary consisting of 4 atoms is considered. As is evident, as the number of atoms in the dictionary increase, fewer iterations will be required to reach the optimal solution.

TABLE I: RESULTS AFTER 1 ITERATION USING COMPLETE DICTIONARY, SPARSITY:1, FEATURES: 10, ATOMS: 10

Sr.	Sparse Coding	Dictionary Update	PSNR (dB)	MSE
1	Normal Equation	Normal Equation	61.122	0.050
2	Normal Equation	K-SVD	51.563	0.453
3	Matching Pursuit	Normal Equation	59.397	0.074
4	Matching Pursuit	K-SVD	48.066	1.014

TABLE II: RESULTS AFTER 50 ITERATIONS USING COMPLETE DICTIONARY, SPARSITY:1, FEATURES: 10, ATOMS: 10

Sr.	Sparse Coding	Dictionary Update	PSNR(dB)	MSE
1	Normal Equation	Normal Equation	61.853	0.042
2	Normal Equation	K-SVD	59.756	0.068
3	Matching Pursuit	Normal Equation	59.325	0.075
4	Matching Pursuit	K-SVD	57.928	0.104

Figures 3 (b-d) represent the reconstructed image and MSE after 1 iteration of our proposed algorithmic framework for cases where recovery is done on 2%, 14% and 23% compressive sensing on the satellite image with different sensing matrices generated randomly. It is observed that the algorithm reaches an almost optimal solution in 1-5 iterations such that any further changes are not easily distinguishable to the human eye. Using blocks of larger window size introduces reconstruction error which highly minimizes as the number of iterations increases. The larger blocks introduce a higher scope to sense compressively. With that using smaller block sizes in our Convex Sparse Dictionary Learning algorithm leads to lower reconstruction MSE. However, using smaller block sizes significant increases the number of signals (blocks) to be sparsely coded. This increases the run-time of the algorithm. The trade-off between run-time, compression ratio, and reconstruction error depends on the application.

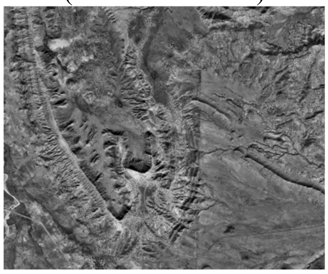
V. CONCLUSIONS

In this article, we have proposed an algorithm for image recovery of compressively sensed satellite images using convex sparse dictionary learning and adaptive spectral filtering. The proposed algorithm works better than other conventional techniques used for the same task, as also, we observe from the results that using Tikhonov regularization for both sparse coding and dictionary learning can be considered as a preferable choice. However, due to the computational complexity of matrix inversion and multiplication, for larger datasets, the proposed method will require significantly larger run-time than the conventional techniques. The compressive sensing technique may further be modified by using variable block sizes depending on the level of detail. Image blocks with low local variance may have larger block sizes sparse coded using a complete dictionary, whereas image blocks exhibiting larger local variance can be learned using smaller block sizes using over-complete dictionaries.

(a) Original Image

(b) Reconstructed Image using 2% samples (MSE = 861.1997)

(c) Reconstructed Image using 14% samples (MSE = 495.4023)



(d) Reconstructed Image using 23% samples (MSE = 357.3654)

Fig. 3. Reconstruted images by the proposed algorithm

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